2012 AMC 12 B, Problem 23.
Corrected Choices, Corrected Solution

February 23, 2012

23. Consider all polynomials of a complex variable, $P(z) = 4z^4 + az^3 + bz^2 + cz + d$, where a, b, c, and d are integers such that $0 \le d \le c \le b \le a \le 4$, and the polynomial has a zero z_0 such that $|z_0| = 1$. What is the sum of all values $P(1)$ over all the polynomials with these properties?

(A) 84 (B) 92 (C) 100 (D) 108 (E) 120

Answer (B): If z_0^k is equal to a positive real r, then $1 = |z_0|^k = |z_0^k| = |r| = r$, so $z_0^k = 1$. Suppose that $z_0^k = 1$. If $k = 1$, then $z_0 = 1$, but $P(1) = 4 + a + b + c + d \ge 4$ so $z_0 = 1$ is not a zero of the polynomial. If $k = 2$, then $z_0 = \pm 1$. If $z_0 = -1$, then $0 = P(-1) = (4 - a) + (b - c) + d$ and by assumption $4 \ge a$, $b \ge c$, and $d \ge 0$. Thus $a = 4$, $b = c$, and $d = 0$. Conversely, if $a = 4$, $b = c$, and $d = 0$, then $P(z) = 4z^4 + 4z^3 + bz^2 + bz = z(z+1)(4z^2 + b)$ satisfies the required conditions. If $k = 3$, then $z_0 = 1$ or $z_0 = \gamma$ where γ is any of the roots of $\gamma^2 + \gamma + 1 = 0$. If $z_0 = \gamma$, then $0 = P(\gamma) = 4\gamma + a + b(-1-\gamma) + c\gamma + d = (a - b) + d + \gamma((4 - b) + c)$ and by assumption $a \ge b$, $d \ge 0$, $4 \ge b$, and $c \ge 0$. Thus $a = b$, $d = 0$, $b = 4$, and $c = 0$. Conversely, if $a = b = 4$ and $c = d = 0$, then $P(z) = 4z^4 + 4z^3 + 4z^2 = 4z^2(z^2 + z + 1)$ satisfies the given conditions because $z_0 = \cos(2\pi/3) + i\sin(2\pi/3)$ is a zero of this polynomial. If $k = 4$, then $z_0 = \pm 1$ or $z_0 = \pm i$. If $z_0 = \pm i$, then $0 = P(\pm i) = 4 \mp ia - b \pm ic + d = (4 - b) + d \mp i(a - c)$ and by assumption $4 \ge b$, $d \ge 0$, and $4 \ge a \ge b \ge c$. Thus $b = 4$, $d = 0$, and $a = c = 4$. Conversely, if $a = b = c = 4$ and $d = 0$, then $P(z) = 4z^4 + 4z^3 + 4z^2 + 4z = 4z(z+1)(z^2+1)$ satisfies the given conditions, but it was already considered in the case when $z_0 = -1$. The remaining case is that z_0^k is not a positive real number for $1 \le k \le 4$. In this case,

$$4z^5 - (z-1)P(z) = z^4(4-a) + z^3(a-b) + z^2(b-c) + z(c-d) + d.$$

If $z = z_0$, then the triangle inequality yields

$$\begin{aligned} 4 &= |z_0^4(4-a) + z_0^3(a-b) + z_0^2(b-c) + z_0(c-d) + d| \\ &\le |z_0^4(4-a)| + |z_0^3(a-b)| + |z_0^2(b-c)| + |z_0(c-d)| + |d| \\ &= |z_0|^4(4-a) + |z_0|^3(a-b) + |z_0|^2(b-c) + |z_0|(c-d) + d \\ &= 4 - a + a - b + b - c + c - d + d = 4. \end{aligned}$$

Thus equality must occur throughout. This means that the vectors $v_4 = z_0^4(4-a)$, $v_3 = z_0^3(a-b)$, $v_2 = z_0^2(b-c)$, $v_1 = z_0(c-d)$, and $v_0 = d$ are parallel and they belong to the same quadrant. If two of these vectors are nonzero, then the quotient must be a positive real number; but dividing the vector with the largest exponent of z_0 by the other would yield a positive rational number times z_0^k for some $1 \le k \le 4$. Because not all of the v_j can be zero, it follows that there is exactly one of them that is nonzero. If $v_0 = d \ne 0$ and $v_1 = v_2 = v_3 = v_4 = 0$, then $4 = a = b = c = d$, and $P(z) = 4z^4 + 4z^3 + 4z^2 + 4z + 4$ satisfies the given conditions because $z_0 = \cos(2\pi/5) + i\sin(2\pi/5)$ is a zero of this polynomial. Finally, if $v_j \ne 0$ for some $1 \le j \le 4$ and the rest are zero, then $4z_0^5 = v_j = z_0^j n$ for some positive integer n, and so $z_0^{5-j} = \frac{1}{4}n$ is a positive real.

Therefore the complete list of polynomials is: $4z^4 + 4z^3 + 4z^2 + 4z + 4$, $4z^4 + 4z^3 + 4z^2$, and $4z^4 + 4z^3 + bz^2 + bz$ with $0 \le b \le 4$. The required sum is $20 + 12 + \sum_{b=0}^{4}(8 + 2b) = 32 + 40 + (2 + 4 + 6 + 8) = 92$.